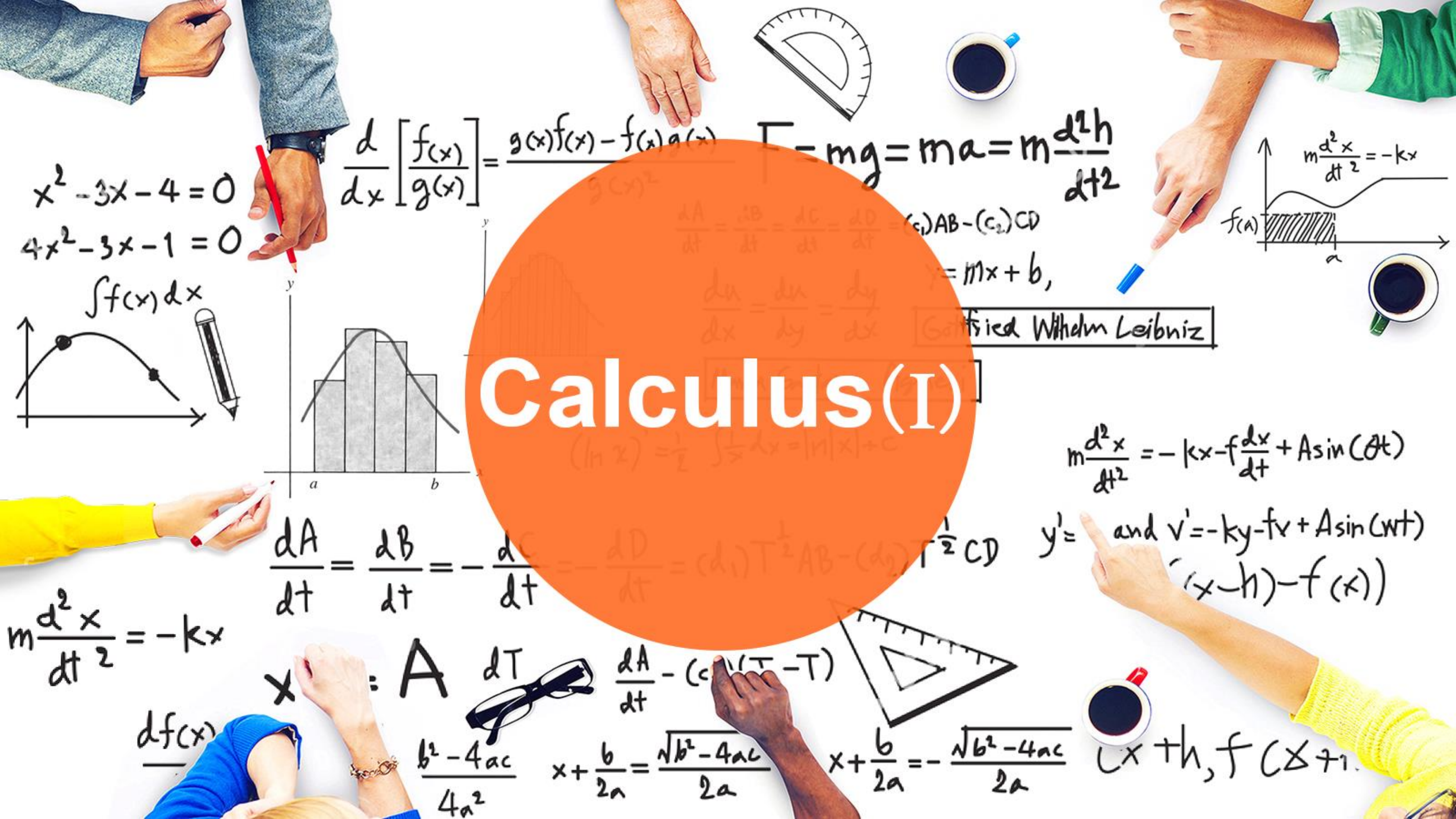

Calculus(Ⅰ)

The Chain Rule

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Problem Introduction

How to find the derivatives of the composite functions?

$$f(x) = \ln \sin x \quad \text{and} \quad f(x) = e^{\sin\frac{1}{x}}$$

Theorem of Chain Rule

Let $y = f(u)$ and $u = g(x)$

If g is differentiable at x and f is differentiable at $u = g(x)$

then the composite function $f \circ g$, defined by $(f \circ g)(x) = f(g(x))$,

is differentiable at x and $(f \circ g)'(x) = f'(g(x))g'(x)$

That is, $D_x f(g(x)) = f'(g(x)) \cdot g'(x)$

or $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Chain Rule

Let $y = f(u)$, $u = \phi(v)$, $v = \psi(x)$,

thus the derivative of $y = f\{\phi[\psi(x)]\}$ is

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Example 1

$f(x) = \ln \sin x$, find $f'(x)$

$$f(x) = y = \ln \sin x$$

Let $\boxed{u = \sin x} \Rightarrow y = \ln u$

$$\because \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore f'(x) = \frac{1}{u} \cdot \cos x = \frac{\cos x}{\sin x} = \cot x.$$

Example 2

$$f(x) = e^{\sin\frac{1}{x}}, \quad \text{find } f'(x)$$

$$f(x) = y = e^{\sin\frac{1}{x}}$$

Let $u = \sin\frac{1}{x}$, $v = \frac{1}{x}$ $\Rightarrow y = e^{u(v)}$

$$\because \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$\therefore f'(x) = e^{u(v)} \cdot \cos v \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2} e^{\sin\frac{1}{x}} \cos\frac{1}{x}.$$

Example 3

$f(x) = (x^2 + 1)^{10}$, find $f'(x)$.

$f(x) = y = (x^2 + 1)^{10}$

Let $u = x^2 + 1 \Rightarrow y = u^{10}$

$$\because \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\therefore f'(x) = 10u^9 \cdot (2x) = 20x(x^2 + 1)^9.$$

Example 4

$$f(x)=\begin{cases}\dfrac{\sin^2 x}{x}, & x\neq 0,\\ 0\quad, & x=0,\end{cases}\quad \text{find}\quad f'(x).$$

$x\neq 0,\quad f'(x)=\left(\dfrac{\sin^2 x}{x}\right)'=\dfrac{\sin 2x}{x}-\dfrac{\sin^2 x}{x^2},$

$x=0,\ f'(0)=\lim\limits_{x\to 0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\dfrac{\dfrac{\sin^2 x}{x}-0}{x}=\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2}=1,$

So, $$f'(x)=\begin{cases}\dfrac{\sin 2x}{x}-\dfrac{\sin^2 x}{x^2}, & x\neq 0,\\ 1\quad, & x=0.\end{cases}$$

Summary of the Chain Rule

The Chain Rules

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

Questions and Answers

$y=\arctan\left(\dfrac{\sin x}{e^x-1}\right)$, find y'.

$$y'=\frac{1}{1+\left(\dfrac{\sin x}{e^x-1}\right)^2}\cdot\left(\frac{\sin x}{e^x-1}\right)'$$

$$=\frac{\cos x(e^x-1)-e^x\sin x}{(e^x-1)^2+\sin^2 x}.$$

Questions and Answers

$$y=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\arcsin\frac{x}{a},\quad \text{find } y'$$

$$y'=\left(\frac{x}{2}\sqrt{a^2-x^2}\right)'+\left(\frac{a^2}{2}\arcsin\frac{x}{a}\right)' \quad (a>0)$$

$$=\frac{1}{2}\sqrt{a^2-x^2}+\frac{x}{2}\cdot\frac{1}{2\sqrt{a^2-x^2}}\cdot(a^2-x^2)'+\frac{a^2}{2}\frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}}\cdot\left(\frac{x}{a}\right)'$$

$$=\frac{1}{2}\sqrt{a^2-x^2}-\frac{x^2}{2\sqrt{a^2-x^2}}+\frac{a^2}{2\sqrt{a^2-x^2}}.$$

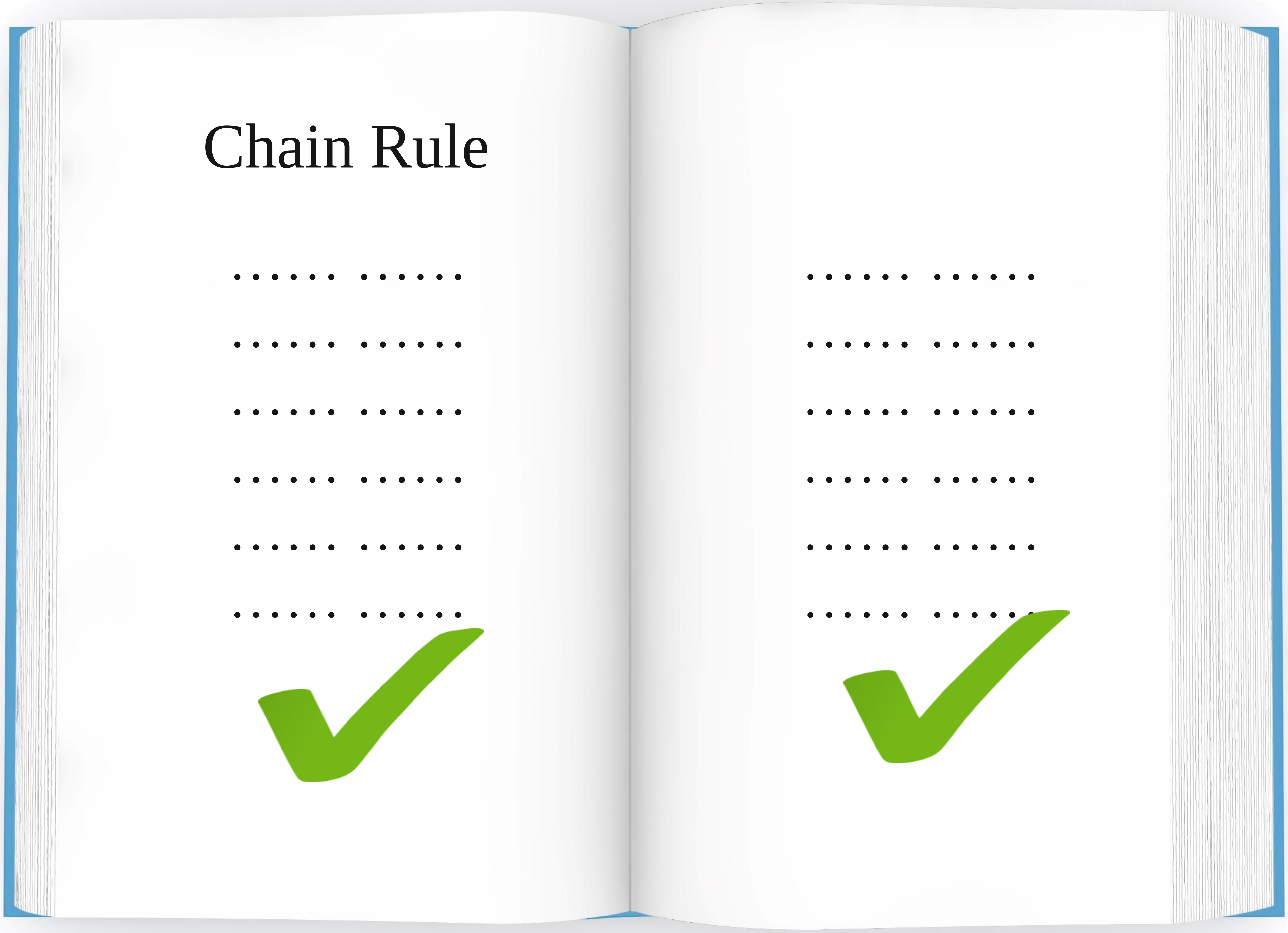
Questions and Answers

$y=\ln\dfrac{\sqrt{x^2+1}}{\sqrt[3]{x-2}}\ (x>2)$, find y'

$\because\ y=\dfrac{1}{2}\ln(x^2+1)-\dfrac{1}{3}\ln(x-2)$

$$\therefore\ y'=\frac{1}{2}\cdot\frac{1}{x^2+1}\cdot 2x-\frac{1}{3(x-2)}$$

$$=\frac{x}{x^2+1}-\frac{1}{3(x-2)}.$$


Chain Rule